second), $(p-1)$ may be filled by 1's in ${}_rC_{p-1}$ ways, and each of these different ways gives rise to $9^{r-(p-1)}$ integers containing the digit 1 p times. Hence, in the first subinterval the number of integers containing 1 at least p times is equal to the sum of expression (1) and ${}_rC_{p-1}\cdot 9^{r-p+1}$. And, therefore, the number of integers in the interval $10^r \leqq x < 10^{r+1}$ which do not contain the digit 1 at least p times, $p \leqq r$, is

$$9\cdot[\text{first } p \text{ terms of expansion of } (9+1)^r] - {}_rC_{p-1}\cdot 9^{r-p+1},$$

or

$$9\cdot\{[\text{first } p \text{ terms of expansion of } (9+1)^r] - {}_rC_{p-1}\cdot 9^{r-p}\}.$$

253. Proposed by HERBERT N. CARLETON, West Newbury, Massachusetts.

Prove that $n^{2k+8} - n^{2k} \equiv 0 \pmod{20}$ for integral values of n and k.

Solution by R. M. Mathews, Riverside, California.

$$n^{2k+8} - n^{2k} = n^{2k}(n^2-1)(n^2+1)(n^4+1)$$

When n is even, $n^{2k} \equiv 0 \pmod 4$. When n is odd, $n^2 - 1 \equiv 0 \pmod 4$.

Next, n being an integer must be of the form $5m$, $5m \pm 1$, or $5m \pm 2$.

For n of the form $5m$, $n^{2k} \equiv 0 \pmod 5$; for n of the form $5m \pm 1$, $n^2 - 1 \equiv 0 \pmod 5$; and for n of the form $5m \pm 2$, $n^2 + 1 \equiv 0 \pmod 5$.

Hence, $n^{2k+8} - n^{2k} \equiv 0 \pmod{20}$, n and k being integers. This is also true of $n^{2k+4} - n^{2k}$.

Also solved by O. S. Adams, W. J. Thome, Elijah Swift, E. B. Escott, C. C. Yen, and the Proposer.

QUESTIONS AND DISCUSSIONS.

Send all communications to U. G. Mitchell, University of Kansas, Lawrence.

REPLIES.

34. Given the mixed integral and functional equation

$$\int_{x=0}^{x=x} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{x}{2}\right) + f(x)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

Remark by S. A. Corey, Albia, Iowa.

The prismoidal formula gives the exact value of this integral whenever the fourth derivative of $f(x) = 0$. This was shown in an article entitled "Certain Integration Formulæ Useful in Numerical Computation" in Vol. XIX, Nos. 6 and 7, of this Monthly, in which formula (1r) is the prismoidal formula including an expression for the remainder term.

$f(x) = Ax^3 + Bx^2 + Cx + D$ is, therefore, the most general value of the function $f(x)$ for which the prismoidal formula gives the exact value for all values of x.

28. Is it possible to obtain $\int \cos\theta^2 d\theta$ without expanding $\cos\theta^2$? If it is not, can some interesting properties of this integral be determined by treating it as a special function?

REPLY BY OSCAR SCHMIEDEL, Bellevue College, Bellevue, Nebraska.

The four reduction formulæ

$$\int \theta^m \sin \theta^n \, d\theta = \frac{1}{m+1} \theta^{m+1} \sin \theta^n - \frac{n}{m+1} \int \theta^{m+n} \cos \theta^n \, d\theta,$$

$$\int \theta^m \sin \theta^n \, d\theta = -\frac{1}{n} \theta^{m+1-n} \cos \theta^n + \frac{m+1-n}{n} \int \theta^{m-n} \cos \theta^n \, d\theta,$$

$$\int \theta^n \cos \theta^n \, d\theta = \frac{1}{m+1} \theta^{m+1} \cos \theta^n + \frac{n}{m+1} \int \theta^{m+n} \sin \theta^n \, d\theta,$$

$$\int \theta^n \cos \theta^n \, d\theta = \frac{1}{n} \theta^{m+1-n} \sin \theta^n - \frac{m+1-n}{n} \int \theta^{m-n} \sin \theta^n \, d\theta,$$

give rise to the following four, which, like those above, give two developments of $\int \theta^m \sin \theta^n d\theta$ and $\int \theta^m \cos \theta^n \, d\theta$ each:

$$\int \theta^m \sin \theta^n \, d\theta = -\frac{\theta^{m+1}}{m+1} \sum_{t=0}^{t=r} n^t \prod_{s=1}^{s=t} \frac{1}{m+sn+1} \cdot \theta^{tn} \sin\left(t\frac{\pi}{2} - \theta^n\right)$$
$$- n^{r+1} \prod_{s=0}^{s=r} \frac{1}{m+sn+1} \cdot \int \theta^{m+(r+1)n} \cos\left(r\frac{\pi}{2} - \theta^n\right) d\theta,$$

$$\int \theta^m \sin \theta^n \, d\theta = \frac{\theta^{m+1}}{m+1} \sum_{t=1-r}^{t=-1} n^t \prod_{s=1+t}^{s=0} (m+sn+1) \cdot \theta^{tn} \sin\left(t\frac{\pi}{2} - \theta^n\right)$$
$$- n^{s-r} \prod_{s=1-r}^{s=-1} (m+sn+1) \cdot \int \theta^{m+(1-r)n} \cos\left(r\frac{\pi}{2} + \theta^n\right) d\theta,$$

$$\int \theta^m \cos \theta^n \, d\theta = \frac{\theta^{m+1}}{m+1} \sum_{t=0}^{t=r} n^t \prod_{s=1}^{s=t} \frac{1}{m+sn+1} \cdot \theta^{tn} \cos\left(t\frac{\pi}{2} - \theta^n\right)$$
$$- n^{1+r} \prod_{s=0}^{s=r} \frac{1}{m+sn+1} \cdot \int \theta^{m+(1+r)n} \sin\left(r\frac{\pi}{2} - \theta^n\right) d\theta,$$

$$\int \theta^m \cos \theta^n \, d\theta = -\frac{\theta^{m+1}}{m+1} \sum_{t=1-r}^{t=-1} n^t \prod_{s=1+t}^{s=0} (m+sn+1) \cdot \theta^{tn} \cos\left(t\frac{\pi}{2} - \theta^n\right)$$
$$+ n^{1-r} \prod_{s=1-r}^{s=-1} (m+sn+1) \cdot \int \theta^{m+(1-r)n} \sin\left(r\frac{\pi}{2} + \theta^n\right) d\theta,$$

where r is everywhere a positive integer.

If in any of these, one integral may be made to vanish, by taking r large enough, then the indicated sum will be the development of the other integral. Thus, making $m = 0$, $n = 2$, in the third of the set, we obtain for the integral of $\cos \theta^2$ the convergent series

$$\int \cos\theta^2\,d\theta = \theta\sum_{t=0} 2^t \prod_{s=1}^{s=t}\frac{1}{2s+1}\theta^{2t}\cos\left(t\frac{\pi}{2}-\theta^2\right)$$

$$= \cos\theta^2\left(\theta - \frac{2^2}{1\cdot3\cdot5}\theta^5 + \frac{2^4}{1\cdot3\cdot5\cdot7\cdot9}\theta^9 - + \cdots\right)$$

$$+ \sin\theta^2\left(\frac{2}{1\cdot3}\theta^3 - \frac{2^3}{1\cdot3\cdot5\cdot7}\theta^7 + - \cdots\right).$$

Two interesting special cases deserve mention.

When the exponent of the coefficient is less by unity than that of the argument of the function, the integral assumes a known form; and the value of the other integral becomes known. Hence, expressions of the type $\theta^{rn-1}\sin\theta^n\,d\theta$ are directly integrable in finite form by these formulæ. If in addition n takes the special form $1/r$, the integral of $\sin\theta^{1/r}d\theta$ is found, that is, the integral of sine or cosine of any root of a variable is found in finite form.

Thus, taking the first of the four formulæ, making $m = n - 1$, and transforming a little; and after this, making $n = 1/r$, we obtain the formulæ:

$$\int \theta^{rn-1}\cos\left(r\frac{\pi}{2}-\theta^n\right)d\theta = -\,n^{-1}\prod_{s=r-1}^{s=1} s\cdot\sum_{t=1}^{t=r}\prod_{s=1}^{s=t-1} s^{-1}\cdot\theta^{(t-1)n}\sin\left(t\frac{\pi}{2}-\theta^n\right),$$

$$\int \cos\left(r\frac{\pi}{2}-\theta^{1/r}\right)d\theta = -\,r\prod_{s=1}^{s=r-1} s\cdot\sum_{t=1}^{t=r}\prod_{s=1}^{s=t-1} s^{-1}\cdot\theta^{(t-1)1/r}\sin\left(t\frac{\pi}{2}-\theta^{1/r}\right).$$

As examples, in these formulas making $n = 2$, $r = 4$, we have

$$\therefore\quad \int \theta^7\cos\theta^2\,d\theta = -\,3\left[\cos\theta^2\left(1-\frac{\theta^4}{2}\right)+\sin\theta^2\left(\theta^2-\frac{\theta^6}{3!}\right)\right],$$

$$\int \cos\theta^{1/4} = -\,4!\left[\cos\theta^{1/4}\left(1-\frac{\theta^{1/2}}{2}\right)+\sin\theta^{1/4}\left(\theta^{1/4}-\frac{\theta^{3/4}}{3!}\right)\right].$$

DISCUSSIONS.

I. Relating to a Curve with Unusual Properties.

By Jos. B. Reynolds, Lehigh University.

The curve

$$y = \frac{a^2x}{x^2+a^2}$$

presents some unusual properties. Considering the part of the curve to the right of the y-axis (see the figure) we have for the area between it and the x-axis

$$A = \int_0^\infty y\,dx = a^2\int_0^\infty \frac{x\,dx}{x^2+a^2} = \frac{a^2}{2}\log(x^2+a^2)\Big]_0^\infty = \infty.$$

Yet the volume generated by revolving this area about the x-axis is

$$V = \pi\int_0^\infty y^2dx = \pi a^4\int_0^\infty \frac{x^2dx}{(x^2+a^2)^2} = \frac{\pi^2a^3}{4};$$